<u>Example</u> Handout by Dr. George Cardwell

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$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = \frac{d}{dt}f(t) + f(t)$$

$$y(0^-) = 2 \,,\quad \dot{y}(0^-) = 1 \qquad f(t) = e^{-4t}u(t)$$

$$\left[s^2Y(s) - s\,y(0^-) - \dot{y}(0^-)\right] + 5\left[sY(s) - y(0^-)\right] + 6Y(s)$$

$$= \left[sF(s) - f(0^-)\right] + F(s)$$

$$(s^2+5s+6)Y(s) - (s+5)y(0^-) - \dot{y}(0^-) = (s+1)F(s) - \cancelto{0}{f(0^-)}$$

$$Y(s) = \frac{s+1}{(s+2)(s+3)}F(s) + \frac{(s+5)\cancelto{2}{y(0^-)} + \cancelto{1}{\dot{y}(0^-)}}{(s+2)(s+3)}$$

$$F(s) = \frac{1}{s+4}$$

$$Y(s) = \frac{s+1}{(s+2)(s+3)(s+4)} + \frac{2s+11}{(s+2)(s+3)}$$

$$= \frac{-1/2}{s+2} + \frac{-2}{s+3} + \frac{-3/2}{s+4} + \frac{7}{s+2} + \frac{-1}{s+3}$$

$$= \frac{13/2}{s+2} - \frac{3}{s+3} - \frac{3/2}{s+4}$$

$$y(t) = \left(\frac{13}{2}e^{-2t} - 3e^{-3t} - \frac{3}{2}e^{-4t}\right)u(t)$$

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As $t \to 0$ from above, that is at $t = 0^+$

$$\boxed{\begin{aligned} y(0^+) &= \left(\frac{13}{2} - \frac{6}{2} - \frac{3}{2}\right) = 2 \\ \dot{y}(0^+) &= \left(-13 + 9 + 6\right) = 2 \end{aligned}}$$

But, the initial conditions at $t = 0^-$ are

$$\boxed{\begin{aligned} y(0^-) &= 2 \\ \dot{y}(0^-) &= 1 \end{aligned}}$$

Apparently, $\dot{y}$ changes discontinuously from 0^- to 0^+. But, it should be discontinuous. The right-hand side contains a δ- function

$$\frac{d}{dt}f(t) + f(t) = -3e^{-4t}u(t) + e^{-4t}\delta(t)$$

So

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \text{Text}\,3e^{-4t}u(t) + e^{-4t}\delta(t)$$

Integrate both sides from 0^- to 0^+. y is continuous, $\frac{dy}{dt}$ is discontinuous but finite.

$$\int_{0^-}^{0^+}\frac{d^2y}{dt^2}dt + 5\cancelto{0}{\int_{0^-}^{0^+}\frac{dy}{dt}dt} + 6\cancelto{0}{\int_{0^-}^{0^+}y\,dt} = \cancelto{0}{\int_{0^-}^{0^+}-3e^{-4t}u(t)dt} + \cancelto{1}{\int_{0^-}^{0^+}e^{-4t}\delta(t)\,dt}$$

$$\boxed{\dot{y}(0^+) - \dot{y}(0^-) = 1}$$

which agrees with what we found above.

Note: $\displaystyle\int_{0^-}^{0^+}-3e^{-4t}u(t)\,dt = \frac{3}{4}e^{-4t}u(t)\Big|_{0^-}^{0^+} - \int_{0^-}^{0^+}\frac{3}{4}e^{-4t}\delta(t)\,dt = 0$